# THREE MANIFOLD GROUPS, KÄHLER GROUPS AND COMPLEX SURFACES

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ABSTRACT. Let $G$ be a Kähler group admitting a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where $N$ is finitely generated.
i) Then $Q$ cannot be non-nilpotent solvable.
ii) Suppose in addition that $Q$ satisfies one of the following:
a) $Q$ admits a discrete faithful non-elementary action on $\mathbb{H}^n$ for some $n \geq 2$
b) $Q$ admits a discrete faithful non-elementary minimal action on a simplicial tree with more than two ends.
c) $Q$ admits a (strong-stable) cut $R$ such that the intersection of all conjugates of $R$ is trivial
Then $G$ is virtually a surface group.

It follows that if $Q$ is infinite, not virtually cyclic, and is the fundamental group of some closed 3-manifold, then $Q$ contains as a finite index subgroup either a finite index subgroup of the 3-dimensional Heisenberg group or the fundamental group of the Cartesian product of a closed oriented surface of positive genus and the circle. As a corollary, we obtain a new proof of a theorem of Dimca and Suciu in [DS09] by taking $N$ to be the trivial group.

If instead, $G$ is the fundamental group of a compact complex surface, and $N$ is finitely presented, then we show that $Q$ must contain the fundamental group of a Seifert-fibered three manifold as a finite index subgroup, and $G$ contains as a finite index subgroup the fundamental group of an elliptic fibration.

We also give an example showing that the relation of quasi-isometry does not preserve Kähler groups. This gives a negative answer to a question of Gromov which asks whether Kähler groups can be characterized by their asymptotic geometry.

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## 1. INTRODUCTION

In this paper, we shall be concerned mainly with the following general set-up:

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1 \qquad\qquad (*)$$

is an exact sequence of finitely generated groups. We shall further assume that $Q$ is infinite and not virtually cyclic. The group $G$ will either be a Kähler group, i.e., the fundamental group of a compact Kähler manifold, or the fundamental group of a compact complex surface. We fix the letters $N, G, i, q$ to have this connotation throughout this paper. We investigate the restrictions that these assumptions impose on the nature of $G$ and $Q$. It will turn out that if in addition, $Q$ is assumed to be the fundamental group of a 3-manifold, then the existence of such an exact sequence $(*)$ with $N$ finitely presented shall force $Q$ to be the fundamental group of a Seifert-fibered 3-manifold.

The following technical Proposition is the starting point of the first part of the paper dealing with Kähler groups. Various restrictions are imposed on $Q$ and we deduce that $G$ is virtually a surface group. Item (i) follows from [De10] [Bru03], Item ii(a) from [DP10], [CT89] and Items ii(b), (c) from the cut-Kähler Theorem of Delzant-Gromov [DG05].

**Proposition 1.1.** *Let $G$ be a Kähler group admitting a short exact sequence*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*where $N$ is finitely generated.*
*i) Then $Q$ cannot be non-nilpotent solvable.*
*ii) Suppose in addition that $Q$ satisfies one of the following:*
*a) $Q$ admits a discrete faithful non-elementary action on $\mathbb{H}^n$ for some $n \geq 2$*
*b) $Q$ admits a discrete faithful non-elementary minimal action on a simplicial tree with more than two ends.*
*c) $Q$ admits a (strong-stable) cut $R$ such that the intersection of all conjugates of $R$ is trivial*
*Then $G$ is virtually a surface group.*

As a consequence we have the following (see Theorem 2.25):

**Theorem 1.2.** *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

*be the exact sequence* (∗) *such that $G$ is a Kähler group and $Q$ is an infinite, not virtually cyclic, fundamental group of some closed 3-manifold. Then there exists a finite index subgroup $Q'$ of $Q$ such that either $Q'$ is a finite index subgroup of the 3-dimensional Heisenberg group or $Q' = \pi_1(\Sigma \times S^1)$, where $\Sigma$ is a closed oriented surface of positive genus.*

(The 3-dimensional Heisenberg group consists of the unipotent upper triangular elements of $GL(3, \mathbb{Z})$.)

Let $Q$ be the fundamental group of a closed 3-manifold which is infinite. Donaldson and Goldman conjectured that $Q$ cannot be a Kähler group. This conjecture was proved by Dimca and Suciu [DS09] and by Delzant (see Theorem 4.3); later a different proof was given by Kotschick [Kot10]. We obtain a proof of this theorem of [DS09] by setting $N$ in Theorem 1.2 to be the trivial group (see Theorem 2.26) and a simple argument to exclude the 3-dimensional Heisenberg group case.

The next theorem deals with the case that $G$ is the fundamental group of a compact complex surface (see Theorem 3.11).

**Theorem 1.3.** *Let*

$$1 \longrightarrow N \stackrel{i}{\longrightarrow} G \stackrel{q}{\longrightarrow} Q \longrightarrow 1$$

*be the exact sequence* (∗) *such that $G$ is the fundamental group of a compact complex surface and $Q$ is an infinite, not virtually cyclic, fundamental group of some closed 3-manifold. Then there exists a finite index subgroup $Q'$ of $Q$ such that $Q'$ is the fundamental group of a Seifert-fibered 3-manifold with hyperbolic or flat base orbifold. Also there exists a finite index subgroup $G'$ of $G$ such that $G'$ is the fundamental group of an elliptic complex surface $X$ which is a circle bundle over a Seifert-fibered 3-manifold.*

In Theorem 1.3, setting $N$ to be the trivial group we conclude that $Q$ is not the fundamental group of a compact complex surface if $Q$ is infinite and not virtually cyclic (see Theorem 3.12).

Stronger results when $X$ is of Class VII, or admits an elliptic fibration, are given in Theorem 3.8 and Proposition 3.10 respectively.

As a consequence of Theorems 1.2 and 1.3 we also get the following result (see Theorem 4.1), the first part of which was proven by J. Hillman [Hi98] based on work of Wall [Wa86]. The second part follows from Theorem 1.2 and the fact that the product of the Heisenberg group with $\mathbb{Z}$ has $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as its abelianization.

**Theorem 1.4.** *Let $M$ be a closed orientable 3 manifold. Then*

*(i) $M \times S^1$ admits a complex structure if and only if $M$ is Seifert fibered.*

*(ii) $M \times S^1$ admits a Kähler complex structure if and only if $M = \Sigma \times S^1$ where $\Sigma$ is a compact surface.*

This can be compared with Thurston's Conjecture, proved by S. Friedl and S. Vidussi [FV09], that $M \times S^1$ admits a symplectic structure if and only if $M$ fibers over a circle. Since a Kähler structure is both symplectic and complex, (ii) above is natural given (i) and the Friedl-Vidussi result.

This paper can be regarded as giving a unified approach generalizing the results of Dimca-Suciu and that of Hillman, by taking $N$ to be trivial in Theorem 1.2 and $N = \mathbb{Z}$ in Theorem 1.3. We would like to point out that the techniques we use are quite different from Dimca-Suciu's as well as Hillman's, which in turn are very different from each other.

Finally, we give an example demonstrating the fact that Kählerness is not preserved by quasi-isometries. This gives a strong negative answer to the following question of Gromov (Problem on page 209 of [Gro93]):

*Can one characterize Kähler groups by their asymptotic geometry?*

The above question makes sense only within the holomorphic category (as the fundamental group of a 2-torus is an index two subgroup of the fundamental group of a Klein bottle). However, if a group $G$ acts freely by holomorphic automorphisms on a simply connected (non-compact) Kähler manifold $X$ and there exists a finite index subgroup $G_1$ of $G$ such that the quotient space $X/G_1$ is Kähler, then $G$ is a Kähler group (cf. [Ara09]).

In the light of this, one makes the above question precise by asking if quasi-isometric to a Kähler group implies commensurable to a (possibly different) Kähler group. This turns out to be false. We give examples of aspherical complex surfaces with quasi-isometric fundamental groups, some of which are Kähler and others not.

The main tools used in this paper are:
(1) A detailed structure theory for 3-manifolds arising from the Geometrization Theorem of Thurston and Perelman.
(2) The cut-Kähler Theorem of Delzant and Gromov [DG05].

In particular, our techniques differ from those of [DS09] and [Kot10]. The main difference is that we approach the problem from a 3-manifolds and Geometric Group Theory perspective rather than a Complex Geometry perspective.

We prove the following key proposition (Case ii(c) of Proposition 1.1 above) which might be of independent interest (See Corollary 2.6 for the proof). We first make the following definition:

Let $Q$ be a group and $R$ a subgroup with the following property: If $M$ is a compact Riemannian manifold with fundamental group $Q$ and $M'$ is the cover of $M$ corresponding to $R$ then

(i) $M'$ has at least 3 ends (i.e., the number of *relative ends* $e(Q, R) \geq 3$) and

(ii) $M'$ is a non-amenable metric space.

We then say that $R$ is a *cut subgroup* of $Q$.

**Proposition 1.5.** *Let*

$$1 \to N \to G \to Q \to 1$$

*be an exact sequence of finitely generated groups. Assume that $G$ is Kähler and $Q$ has a finitely generated cut subgroup $R$. Further assume that no nontrivial normal subgroup of $Q$ is contained in $R$; equivalently assume that the intersection of all conjugates of $R$ is trivial. Then $Q$ is a surface group.*

## 2. Kähler Groups

All manifolds will be connected.

2.1. **Restrictions for Kähler Groups.** In this subsection, we collect together known restrictions on Kähler groups that are used here. See [ABCKT96] for further details.

The following theorem imposes strong restrictions on homomorphisms from a Kähler group to a real hyperbolic lattice.

**Theorem 2.1** ([DP10], [CT89], [Sam86], Section 6.4 of [ABCKT96]). *Let $\Gamma$ be a Kähler group and $\Gamma_1$ be a non-elementary subgroup in $\mathbb{H}^n$, $n \geq 2$. Let $h : \Gamma \longrightarrow \Gamma_1$ be any homomorphism with infinite image. Then $h$ factors through a fibration onto a hyperbolic 2-orbifold.*

The following Corollary establishes Case ii(a) of Proposition 1.1.

**Corollary 2.2.** *Let*
$$1 \longrightarrow N \overset{i}{\longrightarrow} G \overset{q}{\longrightarrow} Q \longrightarrow 1$$
*be an exact sequence of finitely generated groups. Assume that $G$ is Kähler and $Q$ admits a non-elementary action on $\mathbb{H}^n$, $n \geq 2$. Then $G = Q$ is a virtual surface group.*

*Proof.* By Theorem 2.1, $q$ factors through a fibration onto a hyperbolic 2-orbifold $S$, i.e. there exists $h : G \to \pi_1(S)$ and $h_1 : \pi_1(S) \to Q$ such that $h_1 \circ h = q$. In particular $Q = \frac{\pi_1(S)}{h \circ i(N)}$ is the quotient of $\pi_1(S)$ by the finitely generated normal subgroup $h \circ i(N)$. Since $Q$ is infinite, it follows that the finitely generated normal subgroup $h \circ i(N)$ must be trivial (since any finitely generated normal subgroup of the fundamental group of a hyperbolic orbifold is either trivial or of finite index). The Corollary follows. $\square$

The next Theorem imposes strong restrictions on homomorphisms from a Kähler group to solvable groups and establishes Item (i) of Proposition 1.1.

**Theorem 2.3.** [De10] [Bru03] *Let $G$ be a Kähler group such that there is an exact sequence*
$$1 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 1 \,,$$
*where $F$ does not admit a surjective homomorphism onto $F\infty$ (the free group on infinitely many generators), and $H$ is solvable. Then there exist normal subgroups $H_2 \subset H_1$ of $H$ such that*
*(a) $H_1$ has finite index in $H$,*
*(b) $H_1/H_2$ is nilpotent, and*
*(c) $H_2$ is torsion.*

The following version of the *Cut-Kähler Theorem* of Delzant and Gromov is one of the principal tools of this paper. For further details see Sections 6, 7 and 3.8 in [DG05].

**Theorem 2.4.** [DG05] *Let $G$ be the fundamental group $G$ of a compact Kähler manifold $X$, and let $H \subset G$ be a subgroup. Suppose that*

(1) *the cover $X_1$ of $X$ corresponding to the subgroup $H$ is non-amenable, i.e., domains in $X_1$ satisfy a linear isoperimetric inequality, and*
(2) $\#(Ends(X_1)) > 2$.

*Then there is a finite cover $X'$ of $X$ admitting a surjective holomorphic map $X' \longrightarrow S$ to a Riemann surface $S$ with connected fibers, such that the pullback to $\pi_1(X') \subset \pi_1(X)$ of some subgroup in $\pi_1(S)$ coincides with $H \bigcap \pi_1(X')$. In particular, the kernel of the induced homomorphism $\pi_1(X') \longrightarrow \pi_1(S)$ is contained in $H \bigcap \pi_1(X')$.*

**Remark 2.5.** The map from $X$ to $S$ induces a surjection from $\pi_1(X)$ to $\pi_1(S)$. This follows from the well-known fact that a surjective holomorphic map between compact complex manifolds with connected fibers induces a surjection of fundamental groups.

As a corollary of the Cut-Kähler Theorem we have the following which is item ii(c) of Theorem 1.1.

**Corollary 2.6.** *Let*

$$1 \to N \to G \to Q \to 1$$

*be an exact sequence of finitely generated groups. Assume that $G$ is Kähler and $Q$ has a finitely generated cut subgroup $R$. Further assume that no nontrivial normal subgroup of $Q$ is contained in $R$; equivalently suppose that the intersection of all conjugates of $R$ is trivial. Then $Q$ is virtually a surface group.*

Before we prove this, we note the following simple fact:

**Lemma 2.7.** *Let $X$, $Y$ be compact Riemannian manifolds and $q : \pi_1(X) \longrightarrow \pi_1(Y)$ a surjective homomorphism. If $Y_1$ is a cover of $Y$ and $X_1$ the corresponding cover of $X$ (with $\pi_1(X_1) = q^{-1}(\pi_1(Y_1))$), then $X_1$ is quasi-isometric to $Y_1$.*

*Proof.* This follows from the fact that the cover of $X$ corresponding to the subgroup $ker(q_*) \subset \pi_1(X)$ is quasi-isometric to the universal cover of $Y$. $\qquad\square$

*Proof.* (of Corollary 2.6):

Let $X$ be a compact Kähler manifold with fundamental group $G$. Let

$$H := q^{-1}(R) \subset G,$$

and let $X_1$ be the covering of $X$ corresponding to the subgroup $H$. Let $M$ be a compact 2-complex with fundamental group $Q$ and let $M'$ be the cover of $M$ corresponding to $R$. Note that $M'$ has at least 3 ends and is non-amenable by hypothesis. Since $M'$ and $X_1$ are quasi-isometric by Lemma 2.7, the Kähler manifold $X_1$ has these properties as well. We can then apply the cut-Kähler Theorem 2.4 of Delzant-Gromov to conclude that there exists a finite cover $\psi$ from (a finite cover of) $X$ to a Riemann surface $S$ with connected fibers such that the pullback to $\pi_1(X)$ of some subgroup in $\pi_1(S)$ equals $H$. In particular, the kernel

$$ker(\psi_*) \subset H.$$

By hypothesis the normal subgroup $q(ker(\psi_*)) \subset R \subset Q$ must be trivial. Hence $ker(\psi_*) \subset N$. We therefore have an exact sequence

$$1 \longrightarrow \frac{N}{ker(\psi_*)} \longrightarrow \pi_1(S) \longrightarrow Q \longrightarrow 1$$

where $\frac{N}{ker(\psi_*)}$ is a finitely generated normal subgroup of $\pi_1(S)$. Hence $\frac{N}{ker(\psi_*)}$ is trivial and $Q = \pi_1(S)$. $\qquad\square$

**Examples:**

1) Recall the following definition from [GMRS98]: Let $R$ be an infinite subgroup of a group $Q$. We say that the *height* of $R$ in $Q$ is finite if there is a positive integer $n$ such that the intersection of any collection of $n$ distinct conjugates of $R$ corresponding to $n$ distinct coset representatives is trivial. It follows from Corollary 2.6 that if

$$1 \to N \to G \to Q \to 1$$

be an exact sequence of finitely generated groups such that $G$ is Kähler and $Q$ has a finitely generated cut subgroup $R$ of finite height, then $Q$ is virtually a surface group.

2) Let

$$1 \to N \to G \to Q \to 1$$

be an exact sequence of finitely generated groups such that $G$ is Kähler and $Q$ admits a discrete faithful action on a simplicial tree $T$ with more than two ends such that some vertex stabilizer $R$ is finitely generated. The intersection of conjugates of $R$ is trivial as the action is faithful. Also since $T$ has more than two ends, $R$ is a cut subgroup. It follows from Corollary 2.6 that $Q$ is virtually a surface group. This establishes Case ii(b) of Proposition 1.1.

We conclude with a Proposition which might be of independent interest.

**Proposition 2.8.** *Let*

$$1 \to N \to G \to Q \to 1$$

*be an exact sequence of finitely presented groups. Assume that $G$ is Kähler and $Q$ is (Gromov)-hyperbolic. If $Q$ contains a finitely presented quasiconvex codimension one subgroup $H$ (i.e. the number of relative ends $e(Q, H) \geq 2$) then $Q$ must virtually be a surface group.*

*Proof.* An argument in [Gro85] (see also [DG05]) shows that there exists $h \in Q \setminus H$ such that $H$ and $h$ generate the free product $H * < h >= H_1$ which is quasiconvex and $e(Q, H_1) \geq 3$. Therefore $H_1$ is a cut subgroup. Since quasiconvex subgroups have finite height [GMRS98], $H_1$ satisfies the hypotheses of Example 1 above. $\square$

Similar results for relatively hyperbolic groups with codimension one relatively quasiconvex subgroups hold in the light of a result of Hruska-Wise [HW09].

**Example:** A large class of hyperbolic small cancellation groups [Wi04] have codimension one quasiconvex subgroups. It follows that if such a group appears as the quotient group $Q$ in the exact sequence above, it must be virtually a surface group.

2.2. **3 Manifolds.** We very briefly recall the necessary 3-manifold topology and geometry we need and refer to [Hem76], [Bon02], [Sco83a] for details. A closed orientable 3-manifold is said to be prime if it cannot expressed as a non-trivial connected sum (denoted by #) of 3-manifolds. Let $M$ be a closed orientable 3-manifold. Then, there is a unique collection (up to permutation) of prime 3-manifolds $M_1 \cdots, M_k$ such that $M$ is homeomorphic to $M_1 \# \cdots \# M_k$. Further $\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_k)$. Cutting $M$ along essential spheres and capping off the resulting spheres with 3-balls, we get a unique *prime decomposition* of $M$.

Now suppose that $M$ is prime. Then, up to isotopy, there is a unique compact two dimensional submanifold $T$, each of whose components is a 2-sided essential torus, such that every component of $M \setminus T$ either admits a Seifert fibration or else any essential embedded torus in it is parallel to the boundary. Decomposing a 3-manifold along such tori will be called the *torus decomposition*.

A consequence of the Geometrization Theorem (due to Thurston and Perelman), [Per02], [Per03a], [Per03b], is that after prime decomposition and torus decomposition, each resulting piece admits a complete geometric structure modeled on $\mathbb{E}^3$, $S^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, $S^2 \times \mathbb{E}^1$, $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ or *Sol*. Of these, only manifolds modeled on $S^2 \times \mathbb{E}^1$ contains an essential 2-sphere (see [Sco83a]).

**Theorem 2.9.** (*[Bon02], Lemma 2.1 of [KL98]*) *Let $M$ be a closed, orientable prime 3-manifold with nontrivial torus decomposition. Then there is a finite cover $M_1$ of $M$ such that the any piece in the torus decomposition of $M_1$ is either a complete finite volume hyperbolic manifold or of the form $\Sigma \times S^1$, where $S$ is a compact surface with at least two boundary components and of positive genus.*

The following theorem is a consequence of works of Scott, Casson-Jungreis and Gabai as well as Perelman's proof of the Poincaré Conjecture.

**Theorem 2.10** ([CJ94], [Gab92], [Sco83b]). *Let $M$ be a compact, orientable, irreducible 3-manifold such that $\pi_1(M)$ contains an infinite cyclic normal subgroup. Then $M$ is a Seifert-fibered space.*

The next theorem is a consequence of work of Luecke (Theorem 1 of [Lue88]) and the residual finiteness of (geometrizable) 3-manifold groups [Hem87].

**Theorem 2.11.** *Let $M$ be a compact, orientable, irreducible 3-manifold such that at least one of the following conditions is satisfied:*

(a) *$M$ contains an incompressible torus and is not a Sol manifold.*
(b) *$M$ is a Seifert-fibered space such that the base of the Seifert fibration is not an elliptic orbifold.*
(c) *$M$ is not prime.*

*Then $M$ has a finite cover $\rho_1 : M_1 \longrightarrow M$, such that $\operatorname{rank}(H_1(M_1)) \geq 3$. Further, if $M$ is not covered by the 3-torus, then for any positive integer $k$, there is a finite cover, $\rho_k : M_k \longrightarrow M$, such that $\operatorname{rank}(H_1(M_k)) > k$.*

Case (a), where $M$ has a non-trivial torus decomposition, is due to Luecke. Case (b) follows from the fact that fundamental groups of Seifert manifolds admit surjections to the fundamental groups of the associated base orbifold [Sco83a]. Case (c) is a consequence of Grushko's theorem when $M$ is not prime (cf. Lemma 7 of [Kot10]).

**Torus Decompositions and Finite Height** We shall now focus on 3-manifolds admitting a non-trivial torus decomposition and show that if $(\pi_1(M), \pi_1(M_1))$ is a pair such that $M$ is a prime 3-manifold admitting a non-trivial torus decomposition and $M_1$ is one of the pieces, then $\pi_1(M_1)$ **has finite height in** $\pi_1(M)$. In particular, the pair $(\pi_1(M), \pi_1(M_1))$ satisfies the hypotheses on the pair $(Q, R)$ occurring in Corollary 2.6.

A special case where finite height subgroups arise is when groups act *acylindrically* on simplicial trees. Recall [Se97] that a group $Q$ is said to have a *$k$-acylindrical* action on a tree $T$ if the action is minimal and for all $g \in G \setminus \{1\}$, the fixed set of $g$ has diameter bounded by $k$. In this case, it is clear that the height of any vertex stabilizer $R$ is bounded above by $k$.

We shall, in fact, prove the stronger assertion that the action of $\pi_1(M)$ on the Bass-Serre tree associated to the splitting over $\pi_1(M_1)$ is 3-acylindrical. It follows that such an action is faithful and minimal. The proof we give below is due to the referee who gave a substantial simplification of our earlier proof. The proof uses some standard but deep facts from 3-manifold topology, namely the torus and annulus theorems. In the next subsection we shall give a proof that uses coarse geometry only.

**Proposition 2.12.** *Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition and let $T$ be the Bass-Serre tree dual to the torus decomposition of $M$. Then the action of $\pi_1(M)$ on $T$ is 3-acylindrical.*

*Proof.* Suppose $g \in \pi_1(M)(= G)$ fixes a pair of distinct edges in $T$ incident on a common vertex $v \in T$. Then the stabilizer $G_v$ of $v$ is (isomorphic to) the fundamental group of a piece $M_2$ of the torus decomposition of $M$. We claim that $M_2$ must be Seifert-fibered. To see this we first note that $g$ gives rise to an essential annulus in $M_2$, i.e. a map $f : (S^1 \times I, \ S^1 \times \partial I) \to (M_2, \partial M_2)$ such that both $\pi_1(S^1 \times I) \to \pi_1(M_2)$ and $\pi_1(S^1 \times I, \ S^1 \times \partial I) \to \pi_1(M_2, \partial M_2)$ are injections. Next we recall the Annulus Theorem ( [JS79], p. 132, IV 3.2): *Let $N$ be a compact irreducible 3-manifold with $\partial N \neq \phi$. If $N$ contains an essential (immersed) annulus then there exists an essential annulus in $N$ represented by an embedding $g : (S^1 \times I, \ S^1 \times \partial I) \to (N, \partial N)$.*

The existence of an embedded annulus $A \subset M_2$ implies the existence of an essential torus $T_3$ in $M_2$, which is *not* parallel to the boundary. $T_3$ is constructed as follows: Let $C_i \subset \partial M_2$, $i = 1, 2$ be the boundary components (which are circles) of $A$ and let $\nu$ be a sufficiently small open tubular neighborhood of $A_0$ where $A_0 = A \setminus \partial A$. If the boundary tori of $M_2$ containing $C_i$ are $T_i$ then the required essential torus is $T_3 = (T_1 \setminus (T_1 \cap \bar{\nu}) \cup \partial \nu \cup T_2 \setminus (T_2 \cap \bar{\nu}))$ where $\bar{\nu}$ is the closure of $\nu$. Since $T_3$ is not parallel to the boundary, it follows from the Torus Decomposition of 3-manifolds (at the beginning of the present subsection) that $M_2$ is Seifert-fibered.

Hence, if $g$ fixes three consecutive edges $[v_0, v_1]$, $[v_1, v_2]$, $[v_2, v_3]$ of $T$, then the pieces corresponding to $G_{v_1}$ and $G_{v_2}$ must be Seifert-fibered.

One then observes that any essential annulus $A$ in a Seifert-fibered 3-manifold $N$ represents an element of the center $Z(\pi_1(N))$. To see this, note that the presence of an essential annulus implies that the fundamental groups of the corresponding boundary tori intersect. On the other hand, a Seifert fibered space with boundary is a trivial circle bundle over an orbifold and is in fact finitely covered by the product of a surface and a circle [Sco83a]. Hence the intersection of fundamental groups of the two boundary tori is an infinite cyclic group generated by the vertical circle (unless the base orbifold is itself a cylinder $S^1 \times I$). In particular the annulus represents an element of the center of the fundamental group of the Seifert fibered space $N$.

Therefore $g \in Z(G_{v_1}) \cap Z(G_{v_2})$, the intersection of the centers of $G_{v_1}$ and $G_{v_2}$. But $Z(G_{v_1}) \cap Z(G_{v_2}) = \{1\}$ because adjacent Seifert-fibered pieces in the torus decomposition

cannot have homotopic fibers (when each piece is regarded as a trivial circle bundle over an orbifold). Hence $g = 1$.                                                                    □

2.3. **Coboundedness and Finite Height.** In this subsection, we shall give an alternate approach to the content of Proposition 2.12. The purpose of this subsection is to show that if $(\pi_1(M), \pi_1(M_1))$ is a pair such that $M$ is a prime 3-manifold admitting a non-trivial torus decomposition and $M_1$ is one of the pieces, then $\pi_1(M_1)$ **has finite height in** $\pi_1(M)$. In particular, the pair $(\pi_1(M), \pi_1(M_1))$ satisfies the hypotheses on the pair $(Q, R)$ occurring in Corollary 2.6. It follows that the action of such a $\pi_1(M)$ on the Bass-Serre tree associated to the splitting over $\pi_1(M_1)$ is faithful and minimal.

**Structure of Ends:** Let $L_1$ be a non-compact hyperbolic 3-manifold of finite volume, and let $L$ be the manifold with boundary obtained from $L_1$ by removing the interiors of cusps. Let $H_0$ be the torus subgroup of $\pi_1(L)$ corresponding to a cusp. Let $L_T$ denote the cover of $L$ corresponding to the torus subgroup $H_0$. Then $L_T$ is non-amenable; an easy way to see this is to use the fact that $\pi_1(L)$ is strongly hyperbolic relative to the collection of cusp subgroups [Far98].

Similarly, if $\Sigma$ is a compact 2-manifold with boundary and genus greater than one, let $H_0$ denote the cyclic subgroup of $\pi_1(\Sigma)$ corresponding to a boundary component. Then $\pi_1(\Sigma)$ is strongly hyperbolic relative to $H_0$ (cf. Proposition 2.10 of [Mj08]). Hence the cover of $\Sigma$ corresponding to $H_0$ is non-amenable. This implies further that the cover of $\Sigma \times S^1$ corresponding to $H_0 \oplus \mathbb{Z}$ is non-amenable.

Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. By passing to a finite cover, if necessary, and using Theorem 2.9, we can assume that each piece of the torus decomposition of $M$ is either a non-compact complete hyperbolic manifold of finite volume or of the form $\Sigma \times S^1$, with $\Sigma$ a compact surface with non-empty boundary and genus at least two.

Let $M_1$ be a piece of the torus decomposition. Note that each boundary component of $M_1$ is a torus. Then the cover of $M$ corresponding to the subgroup $\pi_1(M_1) \subset \pi_1(M)$ has ends corresponding to the different torus boundary components of $M_1$. Since each end of this cover is non-amenable by the above discussion we have the following lemma.

**Lemma 2.13.** *Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. Let $M_1$ be a piece of the torus decomposition. Then the cover of $M$ corresponding to the subgroup $\pi_1(M_1) \subset \pi_1(M)$ is non-amenable.*

**Coboundedness:**

**Definition 2.14.** *Let $(Y, d)$ be a metric space. We say that two subsets $Y_1, Y_2 \subset Y$ are mutually cobounded if for all $D \geq 0$, the subset*

$$\{(y_1, y_2) \in Y_1 \times Y_2 \mid d(y_1, y_2) \leq D\}$$

*is compact.*

See Section 1 of [MS09] for a closely related definition in the context of hyperbolic metric spaces.

The next two lemmas give standard examples of cobounded subsets of hyperbolic metric spaces (see [Far98], [Bow97] or [Mj06, Section 2.2] for proofs).

We note from Lemma 2.21 below that these examples give groups $Q$ and subgroups $R$ such that no infinite normal subgroup of $Q$ is contained in $R$.

**Lemma 2.15.** *Let $M$ be a non-compact complete hyperbolic n-manifold of finite volume. Let $H_1$ and $H_2$ be lifts of a neighborhood of a cusp in $M$ to the universal cover $\widetilde{M}$. Then $H_1$ and $H_2$ are cobounded horoballs in $\widetilde{M}$. Equivalently, $\partial H_1$ and $\partial H_2$ are cobounded horospheres in $\widetilde{M}$.*

**Lemma 2.16.** *Let $\Sigma$ be a hyperbolic surface with (possibly empty) totally geodesic boundary. Let $\sigma_1$ and $\sigma_2$ be lifts of a closed geodesic in $\Sigma$ to the universal cover $\widetilde{\Sigma}$. Then $\sigma_1$ and $\sigma_2$ are cobounded in $\widetilde{\Sigma}$.*

Now let $N = \Sigma \times S^1$, where $\Sigma$ is an orientable surface with boundary, having genus greater than one. Equip $N$ with a product metric. It follows from Lemma 2.16 that any two lifts $\sigma_1$ and $\sigma_2$ of the boundary components of $\Sigma$ to the universal cover $\widetilde{N}$ are cobounded. Let $E_i = \sigma_i \times \mathbb{R}$ denote the corresponding lift of bounding tori of $N$ to $\widetilde{N}$ containing $\sigma_i$. Thus we have the following:

**Corollary 2.17.** *Let $N = \Sigma \times S^1$, where $\Sigma$ is an orientable surface with boundary, having genus greater than one. Equip $N$ with a product Riemannian metric, and denote by $d_{\widetilde{\Sigma}}$ the induced distance function on $\widetilde{\Sigma}$. Let $\sigma_1$ and $\sigma_2$ be lifts of the boundary components of $\Sigma$ to the universal cover $\widetilde{N}$. Let $E_i = \sigma_i \times \mathbb{R}$, $i = 1, 2$, denote the corresponding lifts of bounding tori of $N$ to $\widetilde{N}$ containing $\sigma_i$. Let $p_i \in \sigma_i$ be such that*

$$d_{\widetilde{\Sigma}}(p_1, p_2) = d_{\widetilde{\Sigma}}(\sigma_1, \sigma_2).$$

*Then for all $D \geq 0$, there exists $\epsilon > 0$ such that*

$$\{(y_1, y_2) \in E_1 \times E_2 | d(y_1, y_2) \leq D\} \subset (B_\epsilon(p_1) \times \mathbb{R}) \times (B_\epsilon(p_2) \times \mathbb{R}),$$

*where $B_\epsilon(p_i)$ denotes the $\epsilon$-neighborhood of $p_i$ in $\sigma_i$.*

Let, as at the beginning of this subsection, $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. Let $M_1$ be a piece of the torus decomposition. Let $\widetilde{M}$ denote the universal cover of $M$ equipped with some equivariant Riemannian metric. The bounding tori of $M_1$ lift to copies of the Euclidean plane in $\widetilde{M}$. Let $\{\mathbb{E}_\alpha\}$ be the collection of lifts (to $\widetilde{M}$) of all tori in the torus decomposition of $M$. If $M_1$ is a hyperbolic piece, then by Lemma 2.15, it follows that any pair $E_1, E_2 \in \{\mathbb{E}_\alpha\}$ which are lifts of bounding tori of $M_1$ is cobounded in $\widetilde{M}$. In fact since the lifts of any hyperbolic piece of the torus decomposition separate $\widetilde{M}$, it follows that there exist $E_1, E_2 \in \{\mathbb{E}_\alpha\}$ separated by such a lift. Hence we have:

**Lemma 2.18.** *Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition such that at least one of the pieces of the torus decomposition is hyperbolic. Let $\{\mathbb{E}_\alpha\}$ be the collection of lifts (to $\widetilde{M}$) of all tori in the torus decomposition of $M$. There exist $E_1, E_2 \in \{\mathbb{E}_\alpha\}$ which are cobounded in $\widetilde{M}$.*

Next assume that $M$ is a prime 3-manifold admitting a non-trivial torus decomposition such that all pieces of $M$ are Seifert-fibered, i.e., $M$ is a graph manifold. By passing to a finite-sheeted cover if necessary we may assume, by Theorem 2.9, that each piece of the

torus decomposition is of the form $\Sigma \times S^1$, where $\Sigma$ is a compact surface with boundary and genus greater than one.

In the universal cover $\widetilde{M}$, let $\widetilde{M}_1$, $\widetilde{M}_2$, $\widetilde{M}_3$, $\widetilde{M}_4$ be a sequence of lifts of Seifert-fibered pieces such that $\widetilde{M}_i \bigcap \widetilde{M}_{i+1} = E_i$, for $i = 1, 2, 3$, where $E_i$ is the universal cover of a torus $T_i$ in $M$. We shall show that $E_1$ and $E_3$ satisfy the conclusions of Lemma 2.18.

Let $M_i = \Sigma_i \times \alpha_i$ where $\alpha_i$ is a unit circle. Assume without loss of generality (by passing to finite-sheeted covers if necessary) that $\{M_i\}_{i=1}^4$ are embedded submanifolds of $M$ and that $M_i \bigcap M_{i+1} = T_i$, for $i = 1, 2, 3$, where $T_i$ is an embedded essential torus in $M$ appearing in the torus decomposition. Also, $T_i = \sigma_i \times \alpha_i$ where $\sigma_i$ is a boundary curve of $\Sigma_i$ and $\alpha_i$ is the circle fiber.

By Corollary 2.17 the set of points $x \in E_1$ and $y \in E_2$ with $d(x, y) \leq D$ must lie within $(B_\epsilon(p_1) \times \widetilde{\alpha_2}) \times (B_\epsilon(p_2) \times \widetilde{\alpha_2})$, where $[p_1, p_2]$ denotes the geodesic in $\widetilde{\Sigma}_2$ joining the appropriate lifts of $\sigma_2$. The same argument shows that the set of points $u \in E_2$ and $v \in E_3$ with $d(u, v) \leq D$ must lie within $(B_\epsilon(q_1) \times \widetilde{\alpha_3}) \times (B_\epsilon(q_2) \times \widetilde{\alpha_3})$, where $[q_1, q_2]$ denotes the geodesic in $\widetilde{\Sigma}_3$ joining the appropriate lifts of $\sigma_3$. Next, $(B_\epsilon(p_2) \times \widetilde{\alpha_2})$ and $(B_\epsilon(q_1) \times \widetilde{\alpha_3})$ are cobounded as the fibers of $M_2$ and $M_3$ do not agree, being different Seifert components. Hence we conclude that $E_1$ and $E_3$ are cobounded in $\widetilde{M}$, in other words, $E_1$ and $E_3$ satisfy the conclusions of Lemma 2.18 as claimed above.

Combining this with Lemma 2.18, we conclude:

**Proposition 2.19.** *Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. Let $\{\mathbb{E}_\alpha\}$ be the collection of lifts (to $\widetilde{M}$) of all tori in the torus decomposition of $M$. There exist $E_1, E_2 \in \{\mathbb{E}_\alpha\}$ which are cobounded in $\widetilde{M}$.*

In fact we have shown the following, which is the main Proposition of this subsection and might be of independent interest.

**Proposition 2.20.** *Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition and let $M_1$ be one of the pieces of the decomposition. Then $\pi_1(M_1)$ has finite height in $\pi_1(M)$.*

Infinite normal subgroups offer the opposite situation, as shown by the following lemma.

**Lemma 2.21.** *Let $M$ be a topological space and $A \subset M$ an incompressible subspace (i.e., $i : A \longrightarrow M$ induces an injective map $i_* : \pi_1(A) \longrightarrow \pi_1(M)$). Suppose $N$ is an infinite normal subgroup of $\pi_1(M)$ contained in $\pi_1(A)$. If $\widetilde{A}$ is a lift of $A$ to $\widetilde{M}$ (the universal cover of $M$), then $g\widetilde{A}$ and $h\widetilde{A}$ are never cobounded for any $g, h \in \pi_1(M)$, where we regard $\pi_1(M)$ as the group of deck transformations of $\widetilde{M}$ and $\widetilde{M}$ has been endowed with a $\pi_1(M)$-equivariant distance function $d$.*

*Proof.* It is enough to prove that $\widetilde{A}$ and $g\widetilde{A}$ are not cobounded for any $g \in \pi_1(M)$. Let $\{t_n\}_{n=1}^\infty \subset N$ be an infinite subset and let $s_n := g t_n g^{-1} \in N$. Fix $x \in \widetilde{A}$. Then $s_n x \in \widetilde{A}$, $g t_n x \in \widetilde{A}$ and $d(s_n x, \ g t_n x) = d(x, gx)$. Since $\{g t_n x\}$ is a noncompact subset of $\widetilde{M}$, we see that $\widetilde{A}$ and $g\widetilde{A}$ are not cobounded by taking $D = d(x, gx)$. $\square$

2.4. **Kähler Groups and 3-Manifold Groups.** In this Section we use the restrictions obtained above to rule out various possibilities. In this section, the following possibilities will be taken up and ruled out one-by-one.

(a) $M$ admits a non-trivial prime decomposition.
(b) $M$ admits a non-trivial torus decomposition.
(c) $M$ admits a *Sol* geometric structure.
(d) $M$ admits a hyperbolic structure.

Spherical or elliptic 3-manifolds have finite fundamental group and 3-manifolds with $S^2 \times \mathbb{R}$ geometry have virtually cyclic fundamental group. These are ruled out by the hypothesis on $Q$. Thus at the end of the discussion we shall conclude that $M$ is a Seifert-fibered space with Euclidean or hyperbolic base orbifold. We shall then proceed to extract further restrictions on what $Q$ may be in case $M$ is Seifert-fibered.

We will use the following proposition for (a) and (b):

We will also need the following simple lemma:

**Lemma 2.22.** *If*

$$1 \longrightarrow N_0 \longrightarrow \pi_1(S) \longrightarrow Q \longrightarrow 1$$

*is an exact sequence of finitely generated groups such that $S$ is a closed orientable surface and $Q$ is an infinite non-virtually cyclic fundamental group of a 3-manifold, then $N_0$ cannot be finitely generated.*

*Proof.* If $S$ has genus $g > 1$ then the only finitely generated normal subgroup of such a $\pi_1(S)$ is the trivial group. This forces $Q$ and $\pi_1(S)$ to be commensurable, an impossibility.

For $S$ a torus, $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}$ and again, such an exact sequence is impossible as it forces $Q$ and $\pi_1(S)$ to be commensurable. $\square$

$M$ **admits a non-trivial prime decomposition**: This follows from Proposition 1.5 and Lemma 2.22 by taking $Q = \pi_1(M)$ and $R = \{1\}$.

$M$ **is prime and admits a non-trivial torus decomposition**: This follows from Proposition 2.12 (or from Proposition 2.20) and Corollary 2.6.

$M$ **is Sol**

Let $G$ be as in $(*)$ such that $Q$ is the fundamental group of a 3-manifold $M$ which is Sol.

Since the group $G$ admits a surjection to the solvable non-nilpotent group $\pi_1(M)$, Theorem 2.3 shows that $G$ cannot be Kähler.

$M$ **hyperbolic**

Let $G$ be as in $(*)$ such that $Q$ is the fundamental group of a 3-manifold $M$ which is hyperbolic.

Since the group $G$ admits a surjection to a closed hyperbolic 3-manifold group $Q$, $G$ cannot be Kähler by Theorem 2.1.

By the above discussion, it follows that if

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

is an exact sequence of groups such that
(a) $Q = \pi_1(M)$ is the fundamental group of a closed 3-manifold $M$,
(b) $Q$ is infinite and not virtually cyclic,
(c) $G$ is Kähler, and
(d) $N$ is finitely presented,
then $M$ is Seifert-fibered with base orbifold Euclidean or hyperbolic.

## $M$ Seifert-fibered

Let $G$ be as in $(*)$ such that $Q$ is the fundamental group of a 3-manifold $M$ which is Seifert-fibered.

Given a surjective homomorphism $h : G \longrightarrow G_1$ with kernel $K$, there is an Euler class obstruction $e(h)$ to the existence of a section of $h$ associated to the exact sequence

$$1 \longrightarrow \frac{K}{[K, K]} \longrightarrow \frac{G}{[K, K]} \longrightarrow G_1 \longrightarrow 1 \,.$$

A recent Theorem of Arapura (Corollary 5.5 of [Ara09]) asserts the following.

**Theorem 2.23.** *If a Kähler group $G$ admits a surjective homomorphism $h$ to a surface group $G_1 = \pi_1(S)$ of genus $g$ greater than one with $g$ maximal, then the Euler class $e(h) \in H^2(G_1, \frac{K}{[K,K]})$ is torsion, where $K$ is the kernel of $h$.*

For a 3-manifold $M$ which is a twisted (non-zero Euler class) circle bundle over a closed hyperbolic orbifold, some cover is a twisted circle bundle over a closed surface $S$ of genus greater than one. By abusing notation slightly we call this cover $M$ and its fundamental group $Q$. Let $G_1 = \pi_1(S)$. If

$$1 \longrightarrow N_0 \, (= \mathbb{Z}) \longrightarrow Q \xrightarrow{\phi} G_1 \longrightarrow 1$$

is the associated exact sequence, then $e(\phi)$ is non-torsion. We have the following.

**Proposition 2.24.** *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

*be an exact sequence of groups such that*

(a) *$Q = \pi_1(M)$ is the fundamental group of a Seifert-fibered closed 3-manifold $M$ with hyperbolic base-orbifold,*
(b) *$G$ is a Kähler group, and*
(c) *$N$ is finitely presented.*

*Then a finite-sheeted cover of $M$ is a product $\Sigma \times S^1$, where $\Sigma$ is a closed oriented surface of genus greater than one.*

*Proof.* If $G$ is Kähler, any finite index subgroup of $G$ is Kähler. Let $G = \pi_1(X)$, where $X$ is a Kähler manifold. By passing to a finite-sheeted cover of $X$ if necessary, we may

assume that $M$ is a circle bundle over a closed surface $S$ of genus greater than one. Abusing notation slightly again, call this cover $X$. We have exact sequences

$$1 \longrightarrow N \xrightarrow{i} \pi_1(X) \xrightarrow{q} Q\,(= \pi_1(M)) \longrightarrow 1$$

and

$$1 \longrightarrow N_0\,(= \mathbb{Z}) \longrightarrow Q \xrightarrow{\phi} G_1\,(= \pi_1(S)) \longrightarrow 1\,.$$

This gives rise to a surjection $\phi \circ q : \pi_1(X) \longrightarrow G_1(= \pi_1(S))$. If $e(\phi)$ is non-zero, it follows that $e(\phi \circ q)$ is non-torsion. Therefore $e(\phi) = 0$ by Theorem 2.23. $\qquad\square$

Hence we have the following, which is one of the main theorems of our paper.

**Theorem 2.25.** *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

*be an exact sequence of groups such that*

(a) $Q = \pi_1(M)$ *is the fundamental group of a closed 3-manifold $M$ such that $Q$ is infinite and not virtually cyclic,*
(b) $G$ *is a Kähler group, and*
(c) $N$ *is finitely presented.*

*Then there exists a finite index subgroup $Q'$ of $Q$ such that either $Q'$ is a finite index subgroup of the 3-dimensional Heisenberg group or $Q' = \pi_1(\Sigma) \times S^1$, where $\Sigma$ is a closed oriented surface of positive genus.*

*Proof.* Proposition 2.24 deals with the case that the base orbifold is hyperbolic. If the base orbifold is Euclidean, then any twisted bundle over the torus has fundamental group isomorphic to (a finite index subgroup of) the 3-dimensional Heisenberg group. $\qquad\square$

If $N$ is trivial and $G$ is a Kähler group, then the exact sequence $(*)$ implies that $Q$ is a Kähler group. But $b_1(\pi_1(\Sigma \times S^1))$ is odd for any closed oriented surface $\Sigma$, hence $\pi_1(\Sigma \times S^1)$ is not Kähler. Also, no finite index subgroup of the 3-dimensional Heisenberg group can be Kähler (Example 3.31 in [ABCKT96, p. 40]). Hence, in view of Theorem 2.25, we have a new proof of the following theorem of Dimca-Suciu (conjectured originally by Donaldson and Goldman).

**Theorem 2.26** ([DS09], [Kot10])**.** *Let $Q$ be the fundamental group of a closed 3-manifold. Suppose further that $Q$ is infinite and not virtually cyclic. Then $Q$ cannot be a Kähler group.*

## 3. Complex Surfaces

3.1. **Restrictions for Complex Non-Kähler Surfaces.** We start with the following theorem due to Kodaira.

**Theorem 3.1** (See Theorem 1.28, Corollary 1.29 of [ABCKT96])**.** *Let $\Gamma$ be a finitely presented group. Then $\Gamma$ is the fundamental group of a Kähler surface if and only if it is the fundamental group of a compact complex surface with even first Betti number.*

We collect together known restrictions on fundamental groups of compact non-Kähler complex surfaces that we will need.

The following Theorem is due to Kodaira; see Theorems 1.27, 1.38 of [ABCKT96], [BHPV04, pp. 244–245], [FM94, Ch. 2]):

**Theorem 3.2.** *Let $X$ be a compact complex surface with odd first Betti number. Then either $X$ is elliptic or of Class VII, i.e., $b_1(X) = 1$ and the Kodaira dimension of $X$ is negative.*

*If $X$ is minimal of Class VII, and admits non-constant meromorphic functions, then $X$ is diffeomorphic to $S^3 \times S^1$.*

*If $X$ is a minimal elliptic surface with odd first Betti number, then the Euler characteristic $\chi(X) = 0$ and the universal cover $\widetilde{X}$ is diffeomorphic to $S^3 \times \mathbb{R}$ or $\mathbb{R}^4$.*

**Corollary 3.3** (Corollary 1.41 of [ABCKT96])**.** *Let $G$ be a group with $b_1(G) = 2m + 1$ with $m \geq 1$ such that $G$ is the fundamental group of a complex surface $X$, Then $X$ is elliptic. If the elliptic fibration $X$ is nonsingular, then $X$ is a $K(G, 1)$ space.*

The next few statements impose restrictions on fundamental groups of class VII complex surfaces.

**Lemma 3.4** (Lemma 1.45 of [ABCKT96])**.** *If $X$ is a minimal complex surface of Class VII, then any finite-sheeted cover of $X$ is of class VII. Further the intersection form on $H_2(X, \mathbb{Z})/\mathrm{Tor}$ is negative definite.*

**Corollary 3.5.** *Let $G$ be a finitely presented group. If $G$ has a finite index subgroup $G'$ such that $b_1(G') > 1$, then $G$ cannot be the fundamental group of a complex surface of Class VII.*

Combining Theorem 2.11 with Corollary 3.5, we have the following:

**Corollary 3.6.** *Let*

$$1 \longrightarrow N \stackrel{i}{\longrightarrow} G \stackrel{q}{\longrightarrow} Q \longrightarrow 1$$

*be an exact sequence of groups, where $Q$ is the fundamental group of a closed 3-manifold $M$, and $N$ is finitely presented. Suppose further that at least one of the following conditions is satisfied:*

  (a) *$M$ contains an incompressible torus and is not a Sol manifold.*
  (b) *$M$ is a Seifert-fibered space such that the base of the Seifert fibration is not an elliptic orbifold.*
  (c) *$M$ is not prime.*

*Then $G$ cannot be the fundamental group of a complex surface of class VII.*

*Proof.* This is a consequence of the following simple fact: Let $A$ and $B$ be topological spaces and $q : \pi_1(A) \longrightarrow \pi_1(B)$ a surjective homomorphism. If $B_1$ is a finite cover of $B$ with positive $b_1$, then the finite cover $A_1$ of $A$ corresponding to the subgroup $q^{-1}(B_1)$ also has positive $b_1$. □

**Theorem 3.7** ([CT97])**.** *Let $M$ be a non-elliptic compact complex surface with first Betti number one and admitting no nonconstant meromorphic functions. Let $N$ be a compact*

*Riemannian manifold of constant negative curvature. Let $\phi : \pi_1(M) \longrightarrow \pi_1(N)$ be a homomorphism. Then the image of $\phi$ is either trivial or an infinite cyclic group.*

### 3.2. **Class VII Surfaces.** Let

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

be an exact sequence of groups where $Q$ is an infinite, non-virtually cyclic fundamental group of a closed 3-manifold $M$. Suppose further that $G$ is the fundamental group of a Class VII surface. *No assumptions are made on N for the purposes of this section.* Then by Corollary 3.6 $M$ is prime and does not admit a non-trivial torus decomposition. Also $M$ cannot be Seifert fibered over a hyperbolic or flat orbifold.

One can assume that $X$ is minimal (as blowing down does not change fundamental group). Then either $X$ is a Hopf surface or else it does not admit any non-constant meromorphic functions.

$X$ cannot be Hopf as Hopf surfaces have infinite cyclic fundamental group and $\pi_1(M)$ is infinite, and not virtually cyclic. Hence $X$ does not admit any non-constant meromorphic functions.

The quotient map from $G$ to $Q$ is surjective. Hence, by Theorem 3.7, the manifold $M$ cannot be hyperbolic.

Finally we dispose of the case that $Q$ is *Sol*. This is ruled out by the following adaptation of an argument of Kotschick [Kot10] that we reproduce here.

Since $\pi_1(X)$ surjects onto $\pi_1(M)$ and the latter fibers over the circle, it follows that the classifying map $\phi_q$ for $q : \pi_1(X) \longrightarrow \pi_1(M)$ induces an isomorphism

$$\phi_{q,1}^* : H^1(M) \longrightarrow H^1(X)$$

and an injective map $\phi_{q,2}^* : H^2(M) \longrightarrow H^2(X)$. Let $\alpha$ be a generator of $H^2(M, \mathbb{Z})$. Then $\alpha \cup \alpha = 0$ and so $\phi_{q,2}^*\alpha \cup \phi_{q,2}^*\alpha = 0$. which makes the intersection form indefinite. This contradicts Lemma 3.4. We summarize our conclusions as follows.

**Theorem 3.8.** *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

*be an exact sequence of groups such that $Q = \pi_1(M)$ is the fundamental group of a closed 3-manifold $M$ which is infinite and not virtually cyclic. Then $G$ cannot be the fundamental group of a Class VII complex surface.*

### 3.3. **Elliptic Fibrations.** Let

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

be an exact sequence of groups where $Q$ is an infinite, non-virtually cyclic fundamental group of a closed 3-manifold $M$. Suppose further that $G$ is the fundamental group of a compact complex surface $X$ admitting an elliptic fibration. Assume that $N$ is finitely generated.

If the fibration is singular, then the inclusion of the fiber subgroup $\mathbb{Z} \oplus \mathbb{Z}$ into $\pi_1(X)$ has non-trivial kernel and $\pi_1(X)$ must have rational cohomological dimension at most 3. In fact, by Theorem 2.3 of [FM94], existence of singular fibers forces $\pi_1(X)$ to be equal

to the fundamental group of the base orbifold of complex dimension one. Hence we have an exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

where $G$ is the fundamental group of an orbifold of complex dimension one and $N$ is a finitely generated normal subgroup of $G$. If the orbifold is hyperbolic, the finitely generated normal subgroup $N$ must be finite and $Q$ must therefore have rational cohomological dimension two. But the assumptions on $Q$ force it to have cohomological dimension 3.

If the orbifold is Euclidean, then after passing to a finite index subgroup if necessary, we have an exact sequence

$$1 \longrightarrow N \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{q} Q \longrightarrow 1 \,.$$

This forces $Q$ to have cohomological dimension at most two. A contradiction again.

Hence the elliptic fibration must be non-singular (with possibly multiple fibers) and we have an exact sequence (after passing to a finite cover again if necessary)

$$1 \longrightarrow N_0 \, (= \mathbb{Z} \oplus \mathbb{Z}) \xrightarrow{\phi} G \xrightarrow{\psi} \pi_1(S) \longrightarrow 1 \,, \tag{1}$$

where $S$ is a closed orientable surface of positive genus. In particular by Corollary 3.3, the group $G$ admits a closed aspherical 4-manifold $X$ as a $K(G, 1)$. Further, there is a holomorphic fiber bundle structure on $X$ realizing the exact sequence (1) with one dimensional complex tori as fibers. The universal cover of $X$ is homeomorphic to $\mathbb{R}^4$.

We first show that $M$ has to be prime. Assume that $M$ is not prime. Then $Q$ is a non-trivial free product that is not virtually cyclic. Hence the abelian normal subgroup $q(N_0) \subset Q$ must be trivial, i.e., $N_0 = \mathbb{Z} \oplus \mathbb{Z}$ must be contained in $N$. Therefore we have an exact sequence

$$1 \longrightarrow \frac{N}{N_0} \longrightarrow \pi_1(S) \longrightarrow Q \longrightarrow 1 \,,$$

where $\frac{N}{N_0}$ is a finitely generated normal subgroup of $\pi_1(S)$, and $Q$ is a non-trivial free product. This is again impossible by Lemma 2.22. Hence $M$ is prime.

Since $\pi_1(M)$ is not virtually cyclic, it follows that $Q$ is a $PD(3)$ group by the sphere theorem. We shall need the following Theorem of Bieri and Eckmann.

**Theorem 3.9** ([BE73]). *Let* $1 \longrightarrow K \longrightarrow H \longrightarrow L \longrightarrow 1$ *be a short exact sequence of groups. If* $K$ *and* $L$ *are duality groups of dimensions* $n$ *and* $m$ *respectively, then* $H$ *is a duality group of dimension* $(m + n)$.

We shall try to understand the structure of $M$ in terms of the rank of $q(N_0)$. The rank of $q(N_0)$ is zero, one, or two.

**Case 1: Rank of $q(N_0)$ is zero.** Then $N \bigcap N_0$ has rank 2 and we have an exact sequence

$$1 \longrightarrow N \bigcap N_0 \longrightarrow G' \xrightarrow{q} Q \longrightarrow 1 \,,$$

where $G'$ is a finitely presented subgroup of $G$. The exact sequence forces $G'$ to have rational cohomological dimension 5 by Theorem 3.9. Since $G$ has cohomological dimension 4, this is impossible.

**Case 2: Rank of $q(N_0)$ is one.** Since the rank of $q(N_0)$ is one, then $Q$ has a normal $\mathbb{Z}$ subgroup. Hence $M$ is a Seifert-fibered space by by Theorem 2.10. Further $N \cap N_0$ has rank one and so there exists an infinite cyclic subgroup $N_1$ of finite index in $N \cap N_0$. Hence we have an exact sequence

$$1 \longrightarrow N_1 \longrightarrow G' \overset{q}{\longrightarrow} Q \longrightarrow 1\,,$$

for some subgroup $G'$ of $G$. This forces $G'$ to be a Poincaré duality group of dimension 4 by Theorem 3.9 and so $G'$ is of finite index in $G$. Hence a finite-sheeted cover of $X$ is a circle bundle over a Seifert-fibered space.

**Case 3: Rank of $q(N_0)$ is two.** Then $Q$ has a normal $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. Hence $M$ is virtually a torus bundle over the circle [Hem76]. Further, $N \cap N_0 = \{1\}$. Hence $NN_0$ is a normal subgroup of $G$ and for all $n \in N, m \in N_0, mn = nm$. Also, $\frac{G}{NN_0} = \mathbb{Z}$. Since $G$ is torsion-free, so is $N$. Also, $N$ must be infinite as $G$ has rational cohomological dimension 4 and $Q$ has rational cohomological dimension 3. Let $H$ be any infinite cyclic subgroup of $N$. Then $HN_0$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and we have a short exact sequence

$$1 \longrightarrow HN_0 \longrightarrow G' \longrightarrow H_1(= \mathbb{Z}) \longrightarrow 1 \tag{2}$$

for some subgroup $G'$ of $G$. This forces $G'$ to be a Poincaré duality group of dimension 4 by Theorem 3.9 and so $G'$ is of finite index in $G$. Further it forces $H$ to have finite index in $N$ and hence $N$ is infinite cyclic. Also the exact sequence in (2) must split and hence by passing to a double cover if necessary we may assume that if $s : H_1 \longrightarrow G'$ is a section, then for all $h \in H$ and $h_1 \in s(H_1)$, $hh_1 = h_1h$. Therefore, $X$ is virtually both a 3-torus bundle over the circle as well as a 2-torus bundle over a 2-torus. By passing to a finite-sheeted cover if necessary, we may assume that $N$ is central. Since each element of $N$ commutes with each element of $N_0$ as well as each element of $s(H_1)$, it follows that $G' = H \oplus Q'$, where $Q'$ is the fundamental group of a 2-torus bundle over the circle.

If $M$ has *Sol* geometry, then $Q'$ is a *Sol* group also and hence $b_1(G') = 1 + b_1(Q') = 2$, which in turn implies that the cover of $X$ with fundamental group $G'$ is Kähler. This contradicts Theorem 2.25. Let $M'$ be the cover of $M$ corresponding to $Q'$. Hence the monodromy of the torus bundle $M'$ is either reducible or periodic. In either case, $M'$ and hence $M$ is Seifert-fibered.

We summarize the discussion as follows.

**Proposition 3.10.** *Let*

$$1 \longrightarrow N \overset{i}{\longrightarrow} G \overset{q}{\longrightarrow} Q \longrightarrow 1$$

*be an exact sequence of groups where $Q = \pi_1(M)$ is an infinite non-virtually cyclic fundamental group of a closed 3-manifold $M$, $G$ is the fundamental group of an elliptic complex surface and $N$ is finitely generated. Then*

(1) *$N$ must be virtually infinite cyclic, and*
(2) *$M$ must be a Seifert-fibered space whose base orbifold is flat or hyperbolic and a finite sheeted cover of $X$ must be a circle bundle over a finite sheeted cover of $M$.*

Combining Proposition 3.10 with Theorem 3.8 we have the second main theorem of our paper.

**Theorem 3.11.** *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

*be an exact sequence of groups such that*

   (a) $Q = \pi_1(M)$ *is the fundamental group of a closed 3-manifold $M$ such that $Q$ is infinite and not virtually cyclic,*
   (b) $G$ *is the fundamental group of a compact complex surface, and*
   (c) $N$ *is finitely presented.*

*Then there exists a finite index subgroup $Q'$ of $Q$ such that $Q'$ is the fundamental group of a Seifert-fibered 3-manifold with hyperbolic or flat base. Also, there exists a finite index subgroup $G'$ of $G$ such that $G'$ is the fundamental group of an elliptic complex surface $X$ which is a circle bundle over a Seifert-fibered 3-manifold.*

Combining Theorem 3.11 with Proposition 3.10 and using the fact that Seifert fibered spaces with hyperbolic or flat base have finite sheeted covers with $b_1 > 1$, we obtain the following theorem of Kotschick.

**Theorem 3.12** ([Kot10]). *Let $Q$ be the fundamental group of a closed 3-manifold. Suppose further that $Q$ is infinite and not virtually cyclic. Then $Q$ cannot be the fundamental group of a compact complex surface.*

## 4. Further Consequences

### 4.1. **Products of 3 Manifolds and Circles.** Let

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

be an exact sequence of groups where $N$ is finitely presented, $Q = \pi_1(M)$ is the fundamental group of a closed Seifert-fibered 3-manifold $M$ whose base orbifold is flat or hyperbolic, and $G$ is the fundamental group of a compact complex surface. By Theorem 3.11, $X$ must be an aspherical elliptic complex surface and $N$ must be virtually infinite cyclic. We now briefly discuss complex structures on circle bundles over $M$, where $M$ is Seifert-fibered.

**Theorem 4.1.** *Let $M$ be a closed orientable 3 manifold. Then $M \times S^1$ admits a complex structure if and only if $M$ is Seifert fibered.*

*Proof.* We will first show that every circle bundle over a Seifert-fibered 3-manifold admits a complex structure (cf. [Wo90]). We include a proof of it for completeness.

Let $X$ be a compact connected Riemann surface, and let $T = \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z})$ be the elliptic curve. Consider the short exact sequence of sheaves on $X$

$$0 \longrightarrow \mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \underline{T} \longrightarrow 0, \tag{3}$$

where $\underline{T}$ is the sheaf of holomorphic functions to $T$, and $\mathcal{O}_X$ is the sheaf of holomorphic functions. Since $H^2(X, \mathcal{O}_X) = 0$ (as $\dim_{\mathbb{C}} X = 1$), the homomorphism

$$H^1(X, \underline{T}) \longrightarrow H^2(X, \mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z})$$

in the long exact sequence of cohomologies associated to the short exact sequence in (3) is surjective. This implies that any $S^1 \times S^1$-bundle over $X$ admits a complex structure.

Let $M \longrightarrow X$ be a Seifert-fibered 3-manifold with (base) orbifold points $\{p_i\}_{i=1}^n$; let $m_i$ be the order of $p_i$. Then a circle bundle over $M$ is diffeomorphic to a manifold obtained by performing logarithmic transformations on a holomorphic principal $T$-bundle over $X$; the logarithmic transformations are done over the points $p_i$ using the automorphism of $T$ given by the automorphism of $\mathbb{C}$ defined by $z \longmapsto z + 1/m_i$. (See [BHPV04, Ch. V, § 13] for logarithmic transformation.) Therefore, we conclude that every circle bundle over a Seifert-fibered surface admits a complex structure.

The converse direction follows from Theorems 2.25, 3.11 and Proposition 3.10.        $\square$

## 4.2. **Quasi-isometries.**

**Theorem 4.2** ([Ger92]). *Two of the eight 3-dimensional geometries, $\widetilde{Sl_2(\mathbb{R})}$ and $\mathbb{H}^2 \times \mathbb{R}$, are quasi-isometric.*

Hence the unit tangent bundle $U(S)$ of a closed surface $S$ of genus greater than one and the product $S \times S^1$ have quasi-isometric fundamental groups. It follows that $H_1 = \pi_1(U(S) \times S^1)$ and $H_2 = \pi_1(S \times S^1 \times S^1)$ are quasi-isometric. Further note that $H_1$ is not commensurable with a Kähler group. This can be seen as follows: If there is a finite extension $K$ of $H_1$ which is Kähler then clearly $H_1$ is Kähler. On the other hand, neither $H_1$ nor any finite index subgroup $H$ of $H_1$ is Kähler: If $X$ is the finite cover of $U(S) \times S^1$ corresponding to $H$ then $X = M \times S^1$ where $M$ is also a nontrivial circle bundle over a surface $S'$. Now a presentation of $\pi_1(M)$ is given by

$$\langle a_1, b_1, \cdots, a_g, b_g, t \mid \Pi_{i=1}^g [a_i, b_i] = t^k \rangle$$

where $\{a_1, b_1, \cdots, a_g, b_g\}$ is the usual set of generators for $\pi_1(S')$ and $k$ is a nonzero integer.

Abelianizing one sees that $b_1(X) = b_1(S') + 1$ and hence $b_1(H)$ is odd. $H_2$ is clearly Kähler being the fundamental group of the product of a torus and an orientable 2-manifold. Thus we have an example demonstrating the fact that Kählerness is not preserved, even up to commensurability, by quasi-isometries within the category of compact aspherical complex surfaces. This gives a strong negative answer to a question of Gromov's (Problem on page 209 of [Gro93]).

## 4.3. **Generalizations and Questions.** After the completion of this paper, Thomas Delzant told us of the following simple and elegant proof of a generalization of Theorem 2.26:

**Theorem 4.3.** *(Delzant) Let $G$ be a Poincaré duality group of dimension three (PD(3) for short) group that does not have property $T$. Then $G$ cannot be Kähler.*

*Proof.* We argue by contradiction. Suppose $G$ is a Kähler group and $M$ is a compact Kähler manifold with fundamental group $G$. One can find a unitary representation $\pi$ with first reduced $l_2$ cohomology $H^1_{red}(G, \pi) \neq 0$. Let $\omega$ be the harmonic representative of such a non-vanishing cohomology class. Then $\omega \wedge \overline{\omega}$ is a closed (1,1) form on the Kähler manifold $M$ which comes from $H^2(G)$. Since $G$ is PD(3), it follows that $H^1(G)$ is not trivial. Hence the Albanese map from $M$ to the Albanese torus is not trivial. The image of this map must be a complex curve; else one would have a non-vanishing cohomology class in $H^4(G)$.

But this curve represent a class in $H^2(G)$ such that every class in $H^1(G)$ has trivial cup product with it, contradicting Poincaré duality. □

One of the conclusions of Theorem 2.25 is that the quotient 3-manifold group $Q$ could be the 3-dimensional Heisenberg group. It is not clear whether this situation can arise at all. Thus we ask

**Question 4.4.** *Do there exist exact sequences*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*with $Q$ the 3-dimensional Heisenberg group, $G$ a Kähler group and $N$ finitely presented (or finitely generated)?*

Question 4.4 seems to be related to the issue of finding which nilpotent groups may arise as Kähler groups. Campana [Cam95] gives examples of nilpotent Kähler groups $H$ fitting into the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow \mathbb{Z}^{\oplus 8} \longrightarrow 1$$

with the presentation $\{t, x_1, y_1, \cdots, x_4, y_4 : [x_i, y_j] = \delta_{ij}t; [x_i, x_j] = 0 = [y_i, y_j]\}$. However, it is not clear which 3-step nilpotent groups may arise as Kähler groups. This seems to be the principal difficulty in addressing Question 4.4 at the moment.

We end with a simple observation based on Campana's example above. It follows from the presentation of the group that the 3-dimensional Heisenberg group can in fact arise as a normal subgroup of a Kähler group. Let $H_1 \subset H$ be the subgroup generated by $\{t, x_1, y_1\}$. Then $H_1$ is isomorphic to the 3-dimensional Heisenberg group. Also the quotient group $H/H_1 = \mathbb{Z}^{\oplus 6}$. Thus we may have non-Kähler finitely presented groups $N$ (here the 3-dimensional Heisenberg group) appearing in an exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with both $G$ and $Q$ Kähler.

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